

VARIATION OF MUMFORD'S QUOTIENTS FOR THE MAXIMAL TORUS ACTION ON A FLAG VARIETY

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ABSTRACT. We study a variation of Mumford's quotient for the action of a maximal torus T on a flag variety G/B depending on projective embedding $G/B \hookrightarrow \mathbb{P}(V(\chi))$, where T -linearization is induced by the standard G -linearization. We describe the linear spans of the supports of semistable orbits, that allow us to calculate the rank of the Picard group of quotient $(G/B)^{ss} // T$, when G does not contain simple factors of type A_n .

Let G be a semisimple algebraic group over an algebraically closed field of characteristic zero, T a maximal torus in G , and B a Borel group containing T . Consider the action of T on G/B by left multiplication. Let χ be a strictly dominant weight. It is well known that G/B can be embedded G -equivariantly in the projectivization $\mathbb{P}(V(\chi))$ of simple module $V(\chi)$ of the highest weight χ as the projectivization of the orbit of the highest weight vector. All G -equivariant embeddings $G/B \hookrightarrow \mathbb{P}^N$ can be obtained by this construction. Denote by L_χ the restriction on G/B of the G -linearized sheaf $\mathcal{O}(1)$ on $\mathbb{P}(V(\chi))$. According to [9], a Zariski open subset $X_{L_\chi}^{ss}$ of the flag variety $X = G/B$ can be defined, in such way that there exists a categorical quotient $X_{L_\chi}^{ss} // T$ for the torus action. In this paper we study variation of the Mumford's quotient depending on the T -linearized sheaf L_χ . We also describe the linear spans of the supports of semistable T -orbits, that allow us to calculate the rank of the Picard group $\text{Pic}(X_{L_\chi}^{ss} // T)$ of the projective variety $X_{L_\chi}^{ss} // T$ (depending on the strictly dominant weight χ). We note that in this case $\text{Pic}(X_{L_\chi}^{ss} // T)$ is finitely generated free abelian group.

For the convenience of the reader we remind the definition of the set of (semi)stable points.

Definition 0.1. Let X be an algebraic variety with an action of G , and L be an invertible ample G -linearized sheaf on X .

(i) Following Mumford we define the set of semistable points as

$$X_L^{ss} = \{x \in X : \exists n > 0, \exists \sigma \in \Gamma(X, L^{\otimes n})^G, \sigma(x) \neq 0\}.$$

(ii) The set of stable points is defined as

$$X_L^s = \{x \in X_L^{ss} : \text{the orbit } Gx \text{ is closed in } X_L^{ss} \text{ and the stabilizer } G_x \text{ is finite}\}.$$

The orbits of maximal torus T on the flag varieties were studied for instance in the papers [2], [3], [12]. In [3] the normality of the closures of typical T -orbits on G/P was proved by R.Dabrowski (where $P \supseteq B$ — is a parabolic subgroup). In [2] the normality of the closures of non-typical T -orbits on G/P was studied by J.B.Carrell and A.Kurth. In [12] S.Senthamarai Kannan has found all the flag varieties G/P with the property that the equality $(G/P)_L^{ss} = (G/P)_L^s$ is satisfied for an invertible sheaf L .

In the first part we derive the Seshadri criterion [13] of the semistability of the point on G/B for the maximal torus action and decompose the Weyl chamber C in the GIT-equivalence classes, for the points with the same sets $X_{L_\chi}^{ss}$. In the second part we describe the linear spans of the supports of the T -orbits of the subvariety

$X_{L_\chi}^{ss}$. In the third part we apply the previous results to the calculation of the rank of $Pic(X_{L_\chi}^{ss} // T)$ when the group G does not contain simple factors of type A_n .

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In the subsequent work the author will consider the case A_n . There is also a hope to study the variation of Mumford's quotient not only for standard, but for all possible T -linearizations of ample line bundles on G/B .

Notations and conventions.

By gothic letters we denote Lie algebras corresponding to Lie groups.

$\Xi = \Xi(T)$ — the lattice of characters of T . Its dual is identified with the lattice of one parameter subgroups

$\Lambda(T)$ by the pairing invariant under the action of Weyl group, that we denote by $\langle \cdot, \cdot \rangle$.

$\Xi_{\mathbb{Q}} = \Xi \otimes_{\mathbb{Z}} \mathbb{Q}$ — rational characters of the torus T .

$\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ — rational one parameter subgroups of the torus T .

$W = N_G(T)/T$ — Weyl group.

On $\Xi_{\mathbb{Q}}$ we have W -invariant scalar product (\cdot, \cdot) , that we use to identify $\Xi_{\mathbb{Q}}$ with $\Lambda_{\mathbb{Q}}$.

Δ is the root system of the Lie algebra \mathfrak{g} corresponding to T . $\Delta^+(\Delta^-)$ is the system of the positive (negative) roots corresponding to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Π is the system of the simple roots. $w_0 \in W$ is the longest element in the Weyl group. C is the positive Weyl chamber.

Let $\tilde{\Delta} \subset \Delta$ be the subset of roots that is an abstract system of roots. $C_{\tilde{\Delta}} := \{\chi \in \Xi_{\mathbb{Q}} | (\chi, \alpha_i) \geq 0 \text{ for all } \alpha_i \in \tilde{\Delta}^+\}$ denotes the positive Weyl chamber for the root system $\tilde{\Delta}$.

We call the root subsystem $\tilde{\Delta} \subset \Delta$ saturated, if the following property holds: $\tilde{\Delta} = \langle \tilde{\Delta} \rangle \cap \Delta$ (here $\langle \tilde{\Delta} \rangle$ denotes the linear span of the root system $\tilde{\Delta}$).

Let $V(\chi)$ be a simple G -module with the highest strictly dominant weight $\chi \in C \cap \Xi$ and the highest weight vector $v_{\chi} \in V(\chi)$.

Consider the action of T on the linear space $V = \bigoplus_{\lambda \in \Xi} V_{\lambda}$ (V_{λ} is the weight component of the weight $\lambda \in \Xi$), $v = \sum_{\lambda \in \Xi} v_{\lambda}$, where $v_{\lambda} \in V_{\lambda}$ and $v_{\lambda} \neq 0$. Denote

by $\text{supp}(v) \subset \Xi_{\mathbb{Q}}$ the convex hull of the weights of vector v . One may notice that $\text{supp}(v) = \text{supp}(tv)$ for every $t \in T$, so the support of a T -orbit is correctly defined, that we denote it by $\text{supp}(Tv)$. We also define the support of the T -invariant subset as the convex hull of the supports of all T -orbits from this set. Let $x = \langle v \rangle \in \mathbb{P}(V)$ be a point corresponding to the vector $v \in V$. Denote by $\text{supp}(Tx \hookrightarrow \mathbb{P}(V))$ the support of the orbit $Tv \hookrightarrow V$. The set of weights of the T -orbit in the embedding $Tx \subset \mathbb{P}(V(\chi))$ is denoted by $Pd_{\chi}(Tx)$.

If it is not stated otherwise, everywhere in the article we consider the flag variety G/B with the G -equivariant embedding in $\mathbb{P}(V(\chi))$ as a close G -orbit of the line, generated by the highest weight vector v_{χ} .

1. A CRITERION OF STABILITY FOR A POINT IN G/B

The stability of the point depends on the Shubert variety to which it belongs. It is essential to study the geometry of the projective embedding for the Schubert variety. Thus we have to remind the lemma from the paper of I.N.Berstein,

I.M.Gelfand, S.I.Gelfand, that describes the structure of the Schubert variety under given projective embedding.

Lemma 1.1. [1, 2.12] *Let $w \in W$ be an element of the Weyl group, BwB/B the corresponding Schubert cell. Consider the closed embedding $G/B \hookrightarrow \mathbb{P}(V(\chi))$, where χ is a strictly dominant weight. Let $f \in V(\chi)$ be a vector from the orbit of the highest weight vector. Then $\langle f \rangle \in BwB/B$ iff $w\chi \in \text{supp}(f)$ and $f \in \mathfrak{U}(\mathfrak{b})v_{w\chi}$, where $\mathfrak{U}(\mathfrak{b})$ is the universal enveloping algebra of the Lie algebra \mathfrak{b} , and $v_{w\chi} = wv_\chi$ is a vector of the weight $w\chi$, entering the irreducible representation $V(\chi)$ with multiplicity one.*

Applying this lemma it is easy to prove the following semistability criterion due to Seshadri [13, Prop. 1.5]. We give the proof that differs from the original one. The ideas similar to those of the proof will be used later. But first we remind the definition of Mumford's numerical function for a torus action and Mumford's numerical criterion of stability.

Definition 1.2. Let L be an ample T -linearized line bundle on a T -variety X defining the T -equivariant embedding of X in the projective space $\mathbb{P}(V)$. Let $\lambda \in \Lambda(T)$ be a one-parameter subgroup. For a point $x \in \mathbb{P}(V)$ we calculate Mumford's numerical function by the formula

$$\mu^L(x, \lambda) = \min_{\tau \in \text{supp}(Tx)} \langle \tau; \lambda \rangle.$$

Proposition 1.3. *Mumford's numerical criterion ([9]) Let X be a variety with a T -action and L be an ample T -linearized line bundle defining T -equivariant embedding X in the projective space $\mathbb{P}(V)$. The point $x \in \mathbb{P}(V)$ is (semi)stable iff $\mu^L(x, \lambda) (\leq) < 0$ for every nontrivial one-parameter subgroup $\lambda \in \Lambda(T)$.*

Proposition 1.4. (Seshadri [13]) *Let C be the Weyl chamber, $x \in G/B$ and $x = bwB/B$. Consider a very ample line bundle L_χ , corresponding to the strictly dominant weight χ . Let $\lambda \in C$ be a one-parameter group belonging to the Weyl chamber. Then we have $\mu^{L_\chi}(x, \lambda) = \langle w\chi; \lambda \rangle$.*

Proof. Consider the support of the orbit Tx . As $Tx \subset BwB/B$, applying the previous lemma, we have $w\chi \in \text{supp}(Tx)$. Also we have $\text{supp}(Tx) \subset \text{supp}(BwB/B) = \text{supp}(\mathfrak{U}(\mathfrak{b})v_{w\chi})$, where the last equality is again the consequence of the previous lemma.

If the weight τ belongs to $\text{supp}(\mathfrak{U}(\mathfrak{b})v_{w\chi})$, then $\tau = w\chi + \sum_{\alpha_i \in \Delta^+} c_i \alpha_i$, where $c_i \geq 0$. Considering the pairing of τ with the one-parameter subgroup λ we get:

$$\langle \tau; \lambda \rangle = \langle w\chi; \lambda \rangle + \sum_{\alpha_i \in \Delta^+} c_i \langle \alpha_i; \lambda \rangle \geq \langle w\chi; \lambda \rangle,$$

as $c_i \geq 0$ and $\langle \lambda; \alpha_i \rangle \geq 0$. Then we may derive the expression for the numerical function:

$$\mu^{L_\chi}(x, \lambda) = \min_{\tau \in \text{supp}(Tx)} \langle \tau; \lambda \rangle = \langle w\chi; \lambda \rangle.$$

□

Now we are ready to get the description of the semistable point set. We need to introduce the following definition.

Definition 1.5. We call $w \in W$ χ -semistable if $\langle w\chi; \lambda \rangle \leq 0$ for every $\lambda \in C$. We define the set of all such elements by W_χ^{st} .

Theorem 1.6. *Consider G -equivariant closed embedding $G/B \hookrightarrow \mathbb{P}(V(\chi))$. Then we can find the set of semistable points of the action of torus (with the linearization coming from the standard T -action on $V(\chi)$) via the following formula:*

$$X_{L_\chi}^{ss} = \bigcap_{\tilde{w} \in W} \bigcup_{w \in W_\chi^{st}} \tilde{w}BwB/B$$

Proof. Indeed, $\bigcup_{w \in W_\chi^{st}} BwB/B$ is the set of such x , that $\mu^{L_\chi}(x, \lambda) \leq 0$ for every one-parameter subgroup $\lambda \in C$.

By the well known equality for numerical functions one has $\mu^{L_\chi}(x, \lambda) = \mu^{L_\chi}(\tilde{w}x, \tilde{w}\lambda)$, where \tilde{w} is a representative of the Weyl group element in the normalizer of T .

From the above we can rewrite the condition of the semistability of x : $\mu^{L_\chi}(x, \lambda) \leq 0$ for every $\lambda \in \Lambda_{\mathbb{Q}}(T)$ as $\mu^{L_\chi}(\tilde{w}x, \lambda) \leq 0$ for every $\lambda \in C$ and $\tilde{w} \in W$. By this we get the following formula for the set satisfying these conditions:

$$X_{L_\chi}^{ss} = \bigcap_{\tilde{w} \in W} \tilde{w} \left(\bigcup_{w \in W_\chi^{st}} BwB/B \right) = \bigcap_{\tilde{w} \in W} \bigcup_{w \in W_\chi^{st}} \tilde{w}BwB/B.$$

□

Remark 1.7. The set $X_{L_\chi}^{ss}$ is the largest $N_G(T)$ -invariant subset in $\bigcup_{w \in W_\chi^{st}} BwB/B$.

Proposition 1.8. *Consider two ample T -linearized line bundles L_{χ_1} and L_{χ_2} . Then $X_{L_{\chi_1}}^{ss} = X_{L_{\chi_2}}^{ss}$ implies $W_{\chi_1}^{st} = W_{\chi_2}^{st}$.*

Proof. It is sufficient to prove that for every element $w \in W_\chi^{st}$ we may find in BwB/B a semistable T -orbit. It will imply that the set W_χ^{st} is uniquely defined by $X_{L_\chi}^{ss}$. We have $\langle w\chi; \lambda \rangle \leq 0$ for every $\lambda \in C$, that is equivalent to $0 \in w\chi + \sum_{\alpha_i \in \Delta^+} \mathbb{Q}_+ \alpha_i$. So by Corollary 2.6 (that we shall prove in the next part because of its technical difficulty) $0 \in \text{supp}(BwB/B)$. For typical orbit Tx from the Schubert cell BwB/B we have $\text{supp}(Tx) = \text{supp}(BwB/B)$, so $0 \in \text{supp}(Tx)$, that means that a typical orbit from the cell BwB/B is semistable. □

We shall decompose the Weyl chamber in cells C_i — GIT-equivalence classes characterized by the following property: two characters χ_1, χ_2 belong to the same cell C_i iff $X_{L_{\chi_1}}^{ss} = X_{L_{\chi_2}}^{ss}$, that is equivalent to $W_{\chi_1}^{st} = W_{\chi_2}^{st}$ by the preceding proposition.

Let A be the cone generated by the simple roots. For an element $\chi \in C^0$ in the relative interior of the Weyl chamber we define the cone $\sigma_\chi = C \cap \bigcap_{\substack{w \in W \\ \chi \in wA}} wA$.

Theorem 1.9. *(Variation of Mumford's quotients)*

The set of the cones $\{\sigma_\chi | \chi \in C^0\}$ is finite. They form the fan with the support C . The internal points of these cones correspond to the GIT-equivalence classes.

Proof. The finiteness of the set of cones follows from the fact, that we get σ_χ as intersections of the cones from the finite set $\{w_i A | w_i \in W\}$. It is easily seen from the definition, that for the cones $\{\sigma_\chi | \chi \in C^0\}$ we have two cases: the face of one cone doesn't contain internal points of the other, one cone is the face of the other (in particular these cones may coincide).

According to the definition, $w \in W_\chi^{st}$ when $\langle w\chi; \lambda \rangle \leq 0$ for every $\lambda \in C$. This is the same as $w\chi \in -A$, which is equivalent to $\chi \in -w^{-1}A$. So the internal points of the cone σ_χ are in one to one correspondence with the elements $\tilde{\chi}$, for which $W_{\tilde{\chi}}^{st} = W_\chi^{st}$. □

Let us begin the study of the set of semistable points. In the subsequent propositions we prove that for semisimple groups not containing simple components of type A_n , the codimension of the set of non-stable points is strictly greater than one.

Lemma 1.10. *Assume that the group G does not contain the simple components of type A_n . Let $s_\alpha \in W$ be the reflection corresponding to a root α . Then $s_\alpha w_0$ belongs to the set W_χ^{st} , for any $\chi \in C^0$.*

Proof. We have to check that $\langle s_\alpha w_0 \chi; \lambda \rangle \leq 0$ for every $\lambda \in C$. Denote by π_α the fundamental weight dual to the simple root α . If $\beta, \gamma \in \Pi$, then $(\pi_\gamma; \beta) = \frac{(\beta; \beta)}{2} \delta_{\gamma\beta}$. Also we have the equality $s_\gamma \pi_\gamma = \pi_\gamma - \gamma$.

The weight χ is dominant, so $w_0 \chi$ is antidominant $w_0 \chi = -\sum c_\beta \pi_\beta$, where $c_\beta > 0$. As $\lambda \in C$ we have $\lambda = \sum a_\beta \pi_\beta$, where $a_\beta \geq 0$. So

$$\begin{aligned} -\langle s_\alpha w_0 \chi; \lambda \rangle &= \langle s_\alpha \sum c_\beta \pi_\beta; \sum a_\beta \pi_\beta \rangle = (c_\alpha (\pi_\alpha - \alpha) + \sum_{\beta \neq \alpha; \beta \in \Pi} c_\beta \pi_\beta; a_\alpha \pi_\alpha + \\ &\sum_{\beta \neq \alpha; \beta \in \Pi} a_\beta \pi_\beta) = c_\alpha a_\alpha ((\pi_\alpha; \pi_\alpha) - \frac{(\alpha; \alpha)}{2}) + (\sum_{\beta \neq \alpha; \beta \in \Pi} c_\beta \pi_\beta; \sum_{\beta \neq \alpha; \beta \in \Pi} a_\beta \pi_\beta) + \\ &(c_\alpha \pi_\alpha; \sum_{\beta \neq \alpha; \beta \in \Pi} a_\beta \pi_\beta) + (\sum_{\beta \neq \alpha; \beta \in \Pi} c_\beta \pi_\beta; a_\alpha \pi_\alpha) \geq 0. \end{aligned}$$

The last equality holds since $(\pi_\alpha; \pi_\alpha) - \frac{(\alpha; \alpha)}{2} \geq 0$ is true for all simple root systems distinct from A_n (that is not difficult to get from Table 2 [10]), other summands are not negative since they are scalar products of two dominant weights. \square

Theorem 1.11. *Let us assume that the group G does not contain the simple components of type A_n . Then the complement to the semistable point set in G/B is of codimension strictly greater than one.*

Proof. We shall use the formula describing the set of semistable points taking into account that, by the previous lemma, $s_\alpha w_0 \in W_\chi^{st}$ for every $\alpha \in \Pi$ and $w_0 \in W_\chi^{st}$:

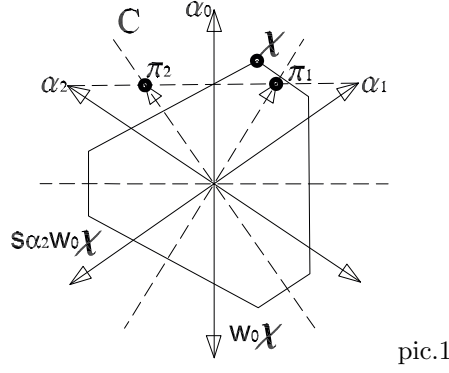
$$X_{L_\chi}^{ss} = \bigcap_{\tilde{w} \in W} \tilde{w} \bigcup_{w \in W_\chi^{st}} BwB/B \supset \bigcap_{\tilde{w} \in W} \tilde{w} \left(\bigcup_{\alpha \in \Pi} Bs_\alpha w_0 B/B \cup Bw_0 B/B \right)$$

The set $\bigcup_{\alpha \in \Pi} Bs_\alpha w_0 B/B \cup Bw_0 B/B$ has already the complement of codimension 2 that implies that $X_{L_\chi}^{ss}$ has the complement of codimension not smaller than 2, as it is equal to the finite intersection of sets with the complement of codimension not smaller than 2 (translation by \tilde{w} does not change the codimension of the complement). \square

Corollary 1.12. *Under the conditions of Theorem 1.11, $\text{Pic}(G/B) = \text{Pic}(X_{L_\chi}^{ss})$ and $\text{Pic}_T(G/B) = \text{Pic}_T(X_{L_\chi}^{ss})$, where $\text{Pic}_T(X)$ denotes the group of T -linearized line bundles on the T -variety X .*

Example 1.13. We give an example which shows that for the root system A_2 the statements of Lemma 1.10 and Theorem 1.11 are not true.

Let $\chi = a\pi_1 + b\pi_2$, where $a > b$ (pic.1). Then $\langle s_{\alpha_2} w_0 \chi; \lambda \rangle > 0$ for $\lambda = \pi_2 \in C$. So $s_{\alpha_2} w_0 \notin W_\chi^{st}$ and the divisor $Bs_{\alpha_2} w_0 B/B$ consists of unstable points.



2. THE LINEAR SPANS OF SUPPORTS FOR SEMISTABLE T -ORBITS

In this part of the work we describe the linear spans of the supports for semistable T -orbits on $X_{L_X}^{ss}$.

Let $x \in G/B$ be written in the form $x = bww_0B/B$. Decompose $b \in B$ into the product of an element of torus and of an unipotent element: $b = tu$. Then $\text{supp}(Tx) = \text{supp}(Tbww_0B/B) = \text{supp}(Tuww_0B/B)$. We represent u as the exponent of an element of the Lie algebra:

$$u = \exp\left(\sum_{\alpha_i \in \bar{\Delta}^+ \subseteq \Delta^+} c_i e_i\right) = \text{id} + \frac{\sum c_i e_i}{1!} + \frac{(\sum c_i e_i)^2}{2!} + \dots,$$

where e_i corresponds to the positive root α_i ; $c_i \neq 0$, and $\bar{\Delta}^+ \subseteq \Delta^+$ is a subset of positive roots. We assume that u is written in the normal form, so that $\bar{\Delta}^+ \subseteq \Delta^+ \cap w\Delta^+$ [7, 28.4].

As $\text{supp}(ww_0B/B) = ww_0\chi$ and the action of T doesn't change the support, from the exponent decomposition for u we get $\text{supp}(Tx) \subset ww_0\chi + \sum_{\alpha_i \in \bar{\Delta}^+} \mathbb{Z}_+\alpha_i$.

We prove now the theorem describing the linear span of the support in terms of the subsystem of positive roots $\bar{\Delta}^+$. We assume that this root system is saturated.

Theorem 2.1. *Let w be an element of the Weyl group such that $ww_0 \in W_\chi^{st}$, and $\bar{\Delta} \subseteq \Delta$ be a saturated subsystem of roots with $0 \in (ww_0\chi + \sum_{\alpha_i \in \bar{\Delta}^+ \cap w\Delta^+} \mathbb{Q}_+\alpha_i)$. Then the support $\text{supp}(Tuww_0B/B)$ of the T -orbit, where $u = \exp(\sum_{\alpha_i \in \bar{\Delta}^+ \cap w\Delta^+} c_i e_i)$, contains zero for almost all c_i .*

Proof. We begin with the following lemma.

Lemma 2.2. *Under the conditions of the previous theorem, let α_0 be the highest root of the root system $\bar{\Delta}$. A strictly dominant weight is contained in the set of weights of the orbit $M \in (-C)^0 \cap \text{supp}(Tuww_0B/B)$. Then the weight $\alpha_0 + M$ is also one of the weights of the T -orbit.*

Proof. Let v_M be the vector of weight M , which belongs to the weight decomposition of the vector corresponding to the point of the orbit Tx . We may describe the vector v_M by the following formula:

$$v_M = \sum_{M=ww_0\chi + \sum_{\alpha_i \in \bar{\Delta}^+} a_i \alpha_i} \frac{c_{\alpha_0}^{a_0} \dots c_{\alpha_l}^{a_l}}{(\sum_i a_i)!} \text{Sym}(e_{\alpha_0}, \dots, e_{\alpha_0}, \dots, e_{\alpha_l}, \dots, e_{\alpha_l}) v_{w_0\chi},$$

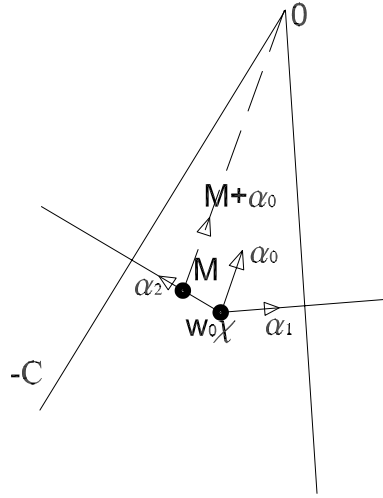
where $Sym()$ denotes the sum over all permutations of the products of the elements in the parentheses.

Note, that the vector v_M is nonzero for almost all values of the coefficients $\{c_{\alpha_i}\}$ iff we may find the set of $\{a_i\}$ for which we have $Sym(\underbrace{e_{\alpha_0}, \dots, e_{\alpha_0}}_{a_0}, \dots, \underbrace{e_{\alpha_l}, \dots, e_{\alpha_l}}_{a_l})v_{w_0\chi} \neq 0$ (we denote this vector by v_M^0). So

to prove that the vector $v_{M+\alpha_0}$ is nonzero (for almost all values of $\{c_{\alpha_i}\}$) it is sufficient to prove that $Sym(e_{\alpha_0}, \underbrace{e_{\alpha_0}, \dots, e_{\alpha_0}}_{a_0}, \dots, \underbrace{e_{\alpha_l}, \dots, e_{\alpha_l}}_{a_l})v_{w_0\chi} \neq 0$.

Let us show that the vector $Sym(e_{\alpha_0}, \underbrace{e_{\alpha_0}, \dots, e_{\alpha_0}}_{a_0}, \dots, \underbrace{e_{\alpha_l}, \dots, e_{\alpha_l}}_{a_l})v_{w_0\chi}$ is proportional to $e_{\alpha_0}v_M^0$. The element e_{α_0} commutes with all e_{α_i} , where $\alpha_i \in \tilde{\Delta}^+$ ($[e_{\alpha_i}, e_{\alpha_0}] \subseteq \mathfrak{g}_{\alpha_i+\alpha_0} = 0$, as the weight $\alpha_i + \alpha_0$ is not the root of the Lie algebra \mathfrak{g}). That means e_{α_0} can be taken out from the sign of symmetrization.

The proof of the lemma will be finished if we show that $e_{\alpha_0}v_M^0 \neq 0$. Consider the representation of the $\mathfrak{sl}_2 \cong \langle e_{\alpha_0}, f_{\alpha_0}, h_{\alpha_0} \rangle$, generated by v_M^0 . Hence $e_{\alpha_0}v_M^0 \neq 0$, as M is strictly antidominant and v_M^0 couldn't be the highest weight vector. \square



pic.2

Remark 2.3. Instead of the claim that the weight M is antidominant we may require a weaker condition: $(M; \alpha_0) < 0$. Then we have $e_{\alpha_0}v_M^0 \neq 0$. Indeed, from the condition $(M; \alpha_0) < 0$ it follows that v_M^0 couldn't be the highest weight vector of the representation of \mathfrak{sl}_2 (the other part of the proof remains the same).

To prove next proposition we need to introduce some additional notation. Denote by M the rational weight and by $\tilde{\Delta}$ a subsystem of roots. Let $H_M(\tilde{\Delta}^+)$ be an affine cone $M + \sum_{\alpha_i \in \tilde{\Delta}^+} \mathbb{Q}_+ \alpha_i$ and $\delta H_M(\tilde{\Delta}^+)$ its border. Sometimes we don't mention the root system in this notation that won't lead to the ambiguity.

Proposition 2.4. *Let $w_0\chi$ is such weight that $0 = w_0\chi + \sum_{\alpha_i \in \tilde{\Delta}^+} c_i \alpha_i$, where $c_i \geq 0$. Then every ray from $(-C)$ with the end in zero point intersects the border $\delta H_{w_0\chi}(\tilde{\Delta}^+)$, and the intersection lies inside the Pd_χ polyhedron of the weights of the representation $V(\chi)$.*

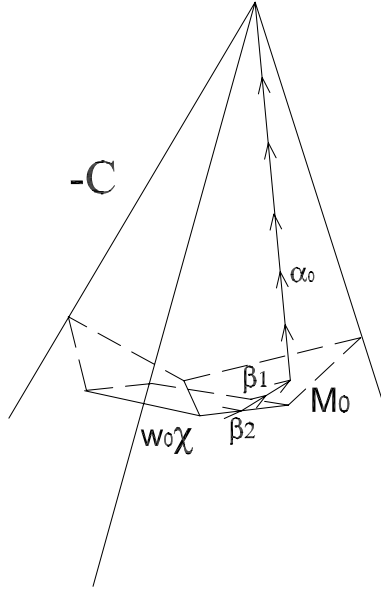
Proof. Consider the ray $l(t) = -t \sum b_i \pi_i$, where $b_i \geq 0$ lying $(-C)$ with the end in zero. We can describe $H_{w_0\chi}$ by the inequalities $(\pi_j, x) \geq (\pi_j, w_0\chi)$. Denote by

Proposition 2.4 it will intersect $\delta H_{ww_0\chi}(\tilde{\Delta}^+) \cap (-C_{\tilde{\Delta}})$ in the point denoted by M_1 . So M_1 belongs to the face of the cone $H_{ww_0\chi}(\tilde{\Delta}^+)$. This face is the cone $H_{ww_0\chi}(\tilde{\Delta}_1^+)$ for a saturated root subsystem $\tilde{\Delta}_1 \subset \tilde{\Delta}$. As M_1 belongs to the antidominant Weyl chamber it also belongs to the chamber $-C_{\tilde{\Delta}_1}$. So we may claim that $M_1 = -k_0\alpha_0$ for $k_0 \in \mathbb{Q}_+$.

Let us describe the induction step. On the i -th step we have the sequence of the saturated root subsystems $\Delta^+ \supseteq \tilde{\Delta}_1^+ \supseteq \dots \supseteq \tilde{\Delta}_{i-1}^+ \supseteq \tilde{\Delta}_i^+$, and also the sequence of the roots $\{\alpha_0, \beta_1, \dots, \beta_i\}$, where β_j is the highest root of the system $\tilde{\Delta}_j$. And we have already constructed the sequence of weights $\{0, M_1, M_2, \dots, M_i\}$ such that $M_j \in H_{ww_0\chi}(\tilde{\Delta}_j^+) \cap (-C_{\tilde{\Delta}_j})$ and also $M_j = M_{j-1} - k_{j-1}\alpha_{j-1}$, where $k_{j-1} \in \mathbb{Q}_+$.

Let us construct the weight M_{i+1} and the root system $\tilde{\Delta}_{i+1}^+$. The weight M_i belongs to the intersection $H_{ww_0\chi}(\tilde{\Delta}_i^+) \cap (-C_{\tilde{\Delta}_i})$. As the highest root β_i lies in the Weyl chamber $C_{\tilde{\Delta}_i}$, the intersection point of the ray $M_i - t\beta_i$ (where $t \in \mathbb{Q}_+$) with the end in M_i and of the border $H_{ww_0\chi}(\tilde{\Delta}_i^+) \cap (-C_{\tilde{\Delta}_i})$ belongs to $\delta H_{ww_0\chi}(\tilde{\Delta}_i^+)$. Denote by $\tilde{\Delta}_{i+1}^+ \subset \tilde{\Delta}_i^+$ the saturated root subsystem corresponding to the face $H_{ww_0\chi}(\tilde{\Delta}_i^+)$ which contains M_i . Observe that $M_{i+1} \in (-C_{\tilde{\Delta}_{i+1}})$ and that the number k_{i+1} is rational as the weight M_i is rational and the cone $H_{ww_0\chi}(\tilde{\Delta}_i^+)$ is rational. Thus all the conditions are satisfied.

Our construction will be finished when $M_i = ww_0\chi$ for some i . All weights M_i are rational so one can find integer N such that all NM_i and also all Nk_i are integer. If we take the weight $Nww_0\chi$ instead of $ww_0\chi$ (this will only change the embedding $G/B \hookrightarrow \mathbb{P}(V(\chi))$ for $G/B \hookrightarrow \mathbb{P}(V(N\chi))$, that don't affect the statement of the theorem). The path constructed above will consist of integer weights NM_i . Also these weights satisfy the equalities $NM_{i+1} = NM_i - \tilde{k}_i\beta_i$, where \tilde{k}_i are integers. So we may suppose that M_i satisfy the conditions described above.



pic.4

We have to show now that all the weights M_i belong to the weights of the orbit $Tuww_0B/B$. In particular that will imply that zero weight lies in the support of the orbit $Tuww_0B/B$. We argue by the induction. One may notice that $ww_0\chi$ belongs to the set of weights $Pd_\chi(Tuww_0B/B)$. Suppose that the weight $M_{i+1} + l\beta_i \in$

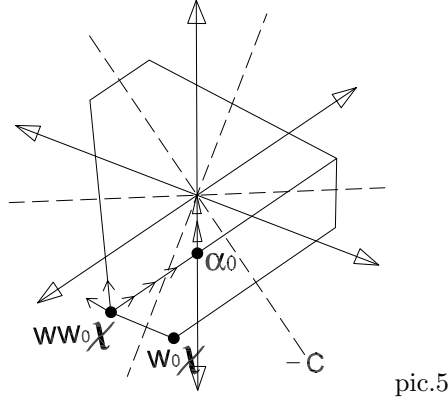
$Pd_\chi(Tuw w_0 B/B)$ for some l , $0 \leq l < k_i$ we must show that $M_{i+1} + (l+1)\beta_i \in Pd_\chi(Tuw w_0 B/B)$.

In order to prove this we apply the sharper variant of Lemma 2.2 (Remark 2.3) to the weight $M_{i+1} + l\beta_i$, root β_i and saturated root system $\tilde{\Delta}_i$. But we have to check that the conditions of the lemma are satisfied: $(M_{i+1} + l\beta_i, \beta_i) < 0$ for every $0 \leq l < k_i$ (*), also we have to show that the root β_i appears in the exponent representing u i.e. $\beta_i \in \tilde{\Delta}^+ \cap w\Delta^+$ (**). In the case $\tilde{\Delta}_i \subsetneq \Delta$ to apply the lemma to the weight $M_{i+1} + l\beta_i$ and the system $\tilde{\Delta}_i$ it is required that the formula for $v_{M_{i+1} + l\beta_i}$ from Lemma 2.2 contains only the terms e_{α_j} with $\alpha_j \in \tilde{\Delta}_i^+$. But this condition is satisfied because $M_{i+1} + l\beta_i \in ww_0\chi + \sum_{\alpha_j \in \tilde{\Delta}_i^+} \mathbb{Q}_+\alpha_j$ and the cone $H_{ww_0\chi}(\tilde{\Delta}_i^+)$ is the face of $H_{ww_0\chi}(\tilde{\Delta})$.

From the construction $M_{i+1} = -(k_i\beta_i + \dots + k_1\beta_1 + k_0\alpha_0)$, where β_j , is the highest root of $\tilde{\Delta}_j^+$. Let β_j be the highest root of the system $\tilde{\Delta}_j^+ \supseteq \tilde{\Delta}_{j+1}^+ \supseteq \dots \supseteq \tilde{\Delta}_i^+$. Then $(\beta_j, \gamma) \geq 0$ for every $\gamma \in \tilde{\Delta}_j^+$, in particular we have $(\beta_j, \beta_m) \geq 0$ for every $m \geq j$. That implies that $(\beta_j, \beta_i) \geq 0$ for every j , and also $(\alpha_0, \beta_i) \geq 0$, as $\beta_i \in \tilde{\Delta}^+$ and α_0 is the highest root of the system $\tilde{\Delta}^+$.

It follows from the above that $(M_{i+1} + l\beta_i, \beta_i) = -((k_i - l)(\beta_i, \beta_i) + \dots + k_1(\beta_1, \beta_i) + k_0(\alpha_0, \beta_i)) < 0$ that proves (*).

Let us check (**). By the construction $-ww_0\chi = (k_l\beta_l + \dots + k_1\beta_1 + k_0\alpha_0)$ for some l . One may notice that $(-w_0\chi, w^{-1}\beta_i) = (-ww_0\chi, \beta_i) = k_i(\beta_l, \beta_i) + \dots + k_1(\beta_1, \beta_i) + k_0(\alpha_0, \beta_i) \geq 0$ as $(\beta_j, \beta_i) \geq 0$ and $(\alpha_0, \beta_i) \geq 0$. From the inequality $(-w_0\chi, w^{-1}\beta_i) \geq 0$ and the fact that the weight $-w_0\chi$ is strictly dominant it follows that the root $w^{-1}\beta_i$ is positive q.e.d. \square



pic.5

Corollary 2.6. *Let χ be a strictly dominant weight and $w \in W$ be an element of Weyl group. Then the following conditions are equivalent:*

- (i) $0 \in ww_0\chi + \sum_{\alpha_i \in \Delta^+} \mathbb{Q}_+\alpha_i$,
- (ii) $0 \in ww_0\chi + \sum_{\alpha_i \in \Delta^+ \cap w\Delta^+} \mathbb{Q}_+\alpha_i$.

These conditions imply that the support of the Schubert cell $Bw w_0 B/B$ contains zero.

Proof. In the proof of the preceding theorem we constructed the piecewise-linear path from $ww_0\chi$ to 0. The roots that appear in that path satisfy the condition $\alpha_i \in \Delta^+ \cap w\Delta^+$. To construct this path we used only the fact that $0 \in ww_0\chi + \sum_{\alpha_i \in \Delta^+} \mathbb{Q}_+\alpha_i$. So the existence of such path implies $0 \in ww_0\chi + \sum_{\alpha_i \in \Delta^+ \cap w\Delta^+} \mathbb{Q}_+\alpha_i$.

Applying the preceding theorem we have $0 \in \text{Supp}(Tuww_0B/B) \subset \text{Supp}(Bww_0B/B)$ for almost all values of c_i (where c_i are the coefficients of e_{α_i} corresponding to the roots $\alpha_i \in \Delta^+ \cap w\Delta^+$ in the exponential decomposition of u). \square

By means of this proposition one can describe the linear spans of the supports for the semistable orbits.

Proposition 2.7. *Let $Tuww_0B/B$ be a semistable orbit from the cell Bww_0B/B . We claim that we can find the orbit $T\tilde{u}ww_0B/B$ from the same cell such that for \tilde{u} we have exponential decomposition $\tilde{u} = \exp(\sum_{\alpha_i \in \tilde{\Delta}^+ \cap w\Delta^+} c_i e_i)$, where the subsystem*

$\tilde{\Delta}$ is saturated and its linear span is the same as the linear span of the support for $Tuww_0B/B$.

Proof. Consider the exponential decomposition $u = \exp(\sum_{\alpha_i \in \tilde{\Delta}^+ \cap w\Delta^+} c_i e_i)$ (now we

do not suppose that $c_i \neq 0$), taking into account that u is chosen in the normal form i.e. $u \in U \cap wU$. Let's consider the cone with the vertex $ww_0\chi$, spanned by the roots $\alpha_i \in \tilde{\Delta}^+ \cap w\Delta^+$ for which $c_i \neq 0$. Let's β_i be the edges of this cone. We show that the vectors $e_{\beta_i}v_{ww_0\chi} \neq 0$ appear in the weight decomposition of the vector from $V(\chi)$ corresponding to the point of the orbit $Tuww_0B/B \subset \mathbb{P}(V(\chi))$.

Let us apply the element w^{-1} to the orbit. As this element lies in $N_G(T)$ we shall get the T -orbit $Tu'w_0B/B$ (where $u' = w^{-1}uw$) and it will belong to the open cell (as $w_0\chi \in \text{supp}(w^{-1}Tuww_0B/B)$ by Lemma 1.1). Besides $\gamma \in \tilde{\Delta}^+ \cap w\Delta^+$ is the root corresponding to e_γ in the exponential decomposition of u iff the root $w^{-1}\gamma \in \Delta^+ \cap w^{-1}\tilde{\Delta}^+$ corresponds to $e_{w^{-1}\gamma}$ in the exponential decomposition of u' . So it is sufficient to prove that $e_{w^{-1}\beta_i}v_{w_0\chi} \neq 0$.

Indeed, the weight $w^{-1}\beta_i \in \Delta^+$, $w_0\chi$ is strictly antidominant so $v_{w_0\chi}$ couldn't be the highest weight vector of the representation of \mathfrak{sl}_2 triple $\langle e_{w^{-1}\beta_i}, f_{w^{-1}\beta_i}, h_{w^{-1}\beta_i} \rangle$ generated by $v_{w_0\chi}$. That gives the claim.

The roots β_i are the edges of the cone consequently the weights $ww_0\chi + \beta_i$ belong to the support of the orbit. Only the vectors $e_{\beta_i}v_{ww_0\chi} \neq 0$ could correspond to them in the weight decomposition (we get this weight decomposition by opening the brackets in the exponential decomposition). The support of the orbit contains zero (the orbit is semistable) so the linear span of the support coincides with the linear span of the roots $\langle \beta_1 \dots \beta_k \rangle_{lin}$. Consider the subsystem of roots $\tilde{\Delta} = \Delta \cap \langle \beta_1 \dots \beta_k \rangle_{lin}$. Applying the preceding theorem to $\tilde{\Delta}$ we get the orbit of special type with the same linear span of the support as the initial orbit has. \square

Thus to describe the linear spans of the supports for all semistable orbits from the cell Bww_0B/B it is sufficient to describe only saturated root systems $\tilde{\Delta}$ for which we have $0 \in (ww_0\chi + \sum_{\alpha_i \in \tilde{\Delta}^+ \cap w\Delta^+} \mathbb{Q}_+ \alpha_i)$.

3. THE CALCULATION OF $\text{Pic}(X^{ss} // T)$.

According to Corollary 1.13, $\text{Pic}_T(X_{L_\chi}^{ss}) = \text{Pic}_T(G/B) \cong \text{Pic}(G/B) \times \Xi(T)$ (the last equality follows from the fact that every line bundle is T -linearized and every two T -linearizations differ by the character).

It is known that $\text{Pic}(G/B) \cong \Xi(B) = \Xi(T)$ [11], [8]. The isomorphism is constructed by the following way: to a line bundle we associate the character of action

of B on the fiber over B -stable point eB/B . And vica versa, to a character χ we may associate the homogenous bundle $G *_B k_\chi$ (k_χ is a linear space, where B acts by the character χ) obtained as a quotient of the $G \times k_\chi$ by B : $b(g, t) = (gb^{-1}, \chi(b)t)$. Under this isomorphism the cone of the very ample line bundles corresponds to the internal points of the Weyl chamber.

T -linearized bundles may be written in the form $G *_B k_{\chi_0} \otimes k_{\chi_1}$. As a line bundle it is isomorphic to $G *_B k_{\chi_0}$, but the torus action is twisted by the character χ_1 .

Let $\pi : X_{L_\chi}^{ss} \longrightarrow X_{L_\chi}^{ss} // T$ be the quotient morphism. It is well known that $\text{Pic}(X_{L_\chi}^{ss} // T)$ injects into $\text{Pic}_T X_{L_\chi}^{ss}$ by the map π^* [8]. In the next theorem we formulate the conditions when the bundle $M \in \text{Pic}_T X_{L_\chi}^{ss}$ belongs to the subgroup $\pi^* \text{Pic}(X_{L_\chi}^{ss} // T)$.

Theorem 3.1. *Let χ be a strictly dominant weight, to which we associate the embedding $G/B \hookrightarrow \mathbb{P}(V(\chi))$. Let $\{\tilde{\Delta}_j^w\}$ be all saturated root subsystems in Δ satisfying the condition $0 \in ww_0\chi + \sum_{\alpha_i \in (\tilde{\Delta}_j^w)^+ \cap w\Delta^+} \mathbb{Q}_+ \alpha_i$. Then the element $\mu = (\mu_0; \mu_1) \in (\text{Pic}(G/B) \times \Xi(T)) \otimes \mathbb{Q}$ belongs to $\pi^* \text{Pic}(X_{L_\chi}^{ss} // T) \otimes \mathbb{Q}$ iff*

$$ww_0\mu_0 + \mu_1 \in \bigcap_j \langle \tilde{\Delta}_j^w \rangle,$$

for all $ww_0 \in W_\chi^{st}$. The rank of the Picard group $\text{Pic}(X_{L_\chi}^{ss} // T)$ is equal to the dimension of the linear space spanned by the points μ satisfying the conditions described above.

Remark 3.2. For the open cell the corresponding condition could be simplified. Let $\{\tilde{\Delta}_j\}$ be all saturated root subsystems such that $w_0\chi \in \langle \tilde{\Delta}_j \rangle$. Then the orbits from the open cell impose the following condition

$$w_0\mu_0 + \mu_1 \in \bigcap_j \langle \tilde{\Delta}_j \rangle,$$

Proof. Consider the following exact sequence from the work [8]:

$$1 \longrightarrow \text{Pic}(X^{ss} // T) \xrightarrow{\pi^*} \text{Pic}_T X^{ss} \xrightarrow{\delta} \prod_{Tx \subset X^{ss}} \Xi(T_x)$$

The last term is the product over all orbits from X^{ss} of groups of characters of the stabilizers of points in these orbits (in the product it is sufficient to consider only the closed orbits of X^{ss}). The map δ is the following: let us take a line bundle $L \in \text{Pic}_T X^{ss}$ and restrict it to the orbit Tx . Then the stabilizer of the point in T_x will act linearly on the fiber over x by a character. This character will be the image of L in the component $\Xi(T_x)$ (denote this map by $Pr_{\Xi(T_x)} : \text{Pic}_T X^{ss} \longrightarrow \Xi(T_x)$), so we have $\delta = \prod_{Tx \subset X^{ss}} Pr_{\Xi(T_x)}$.

Let $\mu = \mu_0 + \mu_1 \in \text{Pic}(G/B) \times \Xi(T)$ be the character defining line bundle L_μ from $\text{Pic}_T X^{ss}$.

Suppose we know $\text{Ker}(Pr_{\Xi(T_x)})$. Then $\text{Pic}(X^{ss} // T) \cong \text{Ker } \delta = \bigcap_{Tx \subset X^{ss}} \text{Ker}(Pr_{\Xi(T_x)})$.

As we calculate the rank of the Picard group, it is sufficient to consider only the action of one-parameter subgroups $\lambda : k^\times \rightarrow T_x$. Let $v_x = v_{\tau_1} + \dots + v_{\tau_l}$ be the weight decomposition of v_x for the embedding $x = twB/B \in G/B \hookrightarrow \mathbb{P}(V(\chi))$, $\tau_i \in \Xi(T)$.

Consider the action of a one-parameter subgroup λ on v_x : $\lambda(t)v_x = t^{\langle \lambda; \tau_0 \rangle} v_x = t^{\langle \lambda; \tau_1 \rangle} v_{\tau_1} + \dots + t^{\langle \lambda; \tau_l \rangle} v_{\tau_l}$. Then $\lambda(k^\times) \subset T_x$ iff $\langle \lambda; \tau_0 \rangle = \langle \lambda; \tau_1 \rangle = \dots = \langle \lambda; \tau_l \rangle$, or equivalently $\langle \lambda; \chi_i - \chi_j \rangle = 0$ for all $\chi_i, \chi_j \in \text{supp}(Tx \hookrightarrow \mathbb{P}(V(\chi)))$. As $0 \in \text{supp}(Tx \hookrightarrow \mathbb{P}(V(\chi)))$ this system of equations is equivalent to $\langle \lambda; \chi_i \rangle = 0$ for $\forall \chi_i \in \text{supp}(Tx)$.

As in the proof of Proposition 2.7 we have $ww_0\chi \in \text{supp}(Tx)$ and also $ww_0\chi + \alpha_i \in \text{supp}(Tx)$, where α_i are the roots appearing in the exponent decomposition of u with the nonzero coefficient. We suppose that u is written in the normal form. Then we have $\langle \lambda; \alpha_i \rangle = 0$, that means that T_x^0 stabilizes u .

Let us calculate the character for the action of T_x^0 on the fiber over x of the bundle $G *_B k_{\mu_0} \otimes k_{\mu_1}$. As T_x^0 stabilizes u , the character for the action of T_x^0 on the fiber over uw_0B/B is equal to the character for the action of T_x^0 on the fiber over ww_0B/B , i.e. $ww_0\mu_0 + \mu_1$.

From the above it follows that the character for the action of T_x^0 on the fiber will be trivial iff $\langle \lambda, ww_0\mu_0 + \mu_1 \rangle = 0$ for every one-parameter subgroup from the stabilizer. So the condition for the line bundle L_μ to be in $\text{Ker}(Pr_{\Xi(T_x)}) \otimes \mathbb{Q}$ can be rewritten in the form $ww_0\mu_0 + \mu_1 \in \langle \text{supp}(Tx \hookrightarrow \mathbb{P}(V(\chi))) \rangle$.

Now we apply Proposition 2.7. The linear spans of the supports for the orbits of special type from $X^{ss} \cap Bww_0B/B$ are in correspondence with the subsystems of positive roots $\{\tilde{\Delta}_j^{w+}\}$, satisfying the condition: $0 \in (ww_0\chi + \sum_{\alpha_i \in (\tilde{\Delta}_j^w)^+ \cap w\Delta^+} \mathbb{Q}\alpha_i)$.

Rewrite the conditions $\mu \in \text{Ker}(\delta)$ in terms of the weight χ :

$$ww_0\mu_0 + \mu_1 \in \bigcap_j \langle \tilde{\Delta}_j^w \rangle.$$

□

Example 3.3. Consider the case of general position; i.e., when the character $w_0\chi$ does not belong to linear spans of root subsystems in Δ , those dimension is less than the rank of the system Δ . In this case there are no conditions both on μ_0 and μ_1 , so we have $\text{rk}(\text{Pic}(X_{L_\chi}^{ss} // T)) = \text{rk}(\text{Pic}_T(X_{L_\chi}^{ss})) = 2\text{rk}G$ (in the case free of the components of type A_n).

Example 3.4. Consider the root system B_4 . The simple roots and the fundamental weights dual to them are the following: $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_4$, and $\pi_1 = \varepsilon_1, \pi_2 = \varepsilon_1 + \varepsilon_2, \pi_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \pi_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. Consider the strictly dominant weight $\chi = (\varepsilon_4 - \varepsilon_3) + 10(2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = 10\pi_1 + \pi_2 + 8\pi_3 + 2\pi_4$. One may check that the only subspaces that contain this element are the linear spans of the following root systems: the first is Δ_1 of type A_3 , generated by the roots $\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_1 + \varepsilon_4$, which are the simple roots of this system (it also contain the roots $\varepsilon_2 - \varepsilon_4, \varepsilon_1 + \varepsilon_3, \varepsilon_1 + \varepsilon_2$ as the positive roots); the second is Δ_2 of type $A_1 \oplus A_2$, generated by $\varepsilon_1, \varepsilon_2 + \varepsilon_4, \varepsilon_3 - \varepsilon_4$, which are the simple roots in it (it also contains $\varepsilon_2 + \varepsilon_3$ as the positive root). The intersection is two dimensional space generated by the root $\varepsilon_3 - \varepsilon_4$ and the dominant weight $2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \pi_1 + \pi_3$.

We show that $\text{rk}(\text{Pic}(X_{L_\chi}^{ss} // T)) = 2$.

Let us write down the conditions of Theorem 3.1 for the part of the χ -semistable elements $w_0, s_{\alpha_1}w_0, s_{\alpha_2}w_0, s_{\alpha_3}w_0, s_{\alpha_4}w_0$ of the Weyl group:

$$\begin{aligned} w_0\mu_0 + \mu_1 &\in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle \\ w_0\mu_0 + s_{\alpha_1}\mu_1 &\in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle \\ w_0\mu_0 + s_{\alpha_2}\mu_1 &\in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle \end{aligned}$$

$$w_0\mu_0 + s_{\alpha_3}\mu_1 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$$

$$w_0\mu_0 + s_{\alpha_4}\mu_1 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$$

To prove the last four expressions one has to check that $(s_{\alpha_i}\chi \in \sum_{\beta_j \in (s_{\alpha_i}\Delta_1)^+ \cap s_{\alpha_i}\Delta^+} \mathbb{Q}_+\beta_j)$ and the same for the system Δ_2 . But $(s_{\alpha_i}\Delta_1)^+ \cap s_{\alpha_i}\Delta^+ = s_{\alpha_i}\Delta_1 \cap s_{\alpha_i}\Delta^+ \cap \Delta^+$, so if we apply s_{α_i} to two last expressions we get $(\chi \in \sum_{\beta_j \in \Delta_1^+ \setminus \{\alpha_i\}} \mathbb{Q}_+\beta_j)$. Indeed s_{α_i} is a simple reflection so $\Delta^+ \cap s_{\alpha_i}\Delta^+ = \Delta^+ \setminus \{\alpha_i\}$, that implies $s_{\alpha_i}(s_{\alpha_i}\Delta_1 \cap s_{\alpha_i}\Delta^+ \cap \Delta^+) = (\Delta_1^+ \setminus \{\alpha_i\})$

We note that in the case $i = 1, 4$ the condition is satisfied automatically since the roots α_1, α_4 do not belong to the systems $\Delta_1 \Delta_2$. In the case $i = 2$ the second condition is satisfied since $\alpha_2 \notin \Delta_2$, and for the first system it is satisfied since $\chi = 10(\varepsilon_1 + \varepsilon_2) + 9(\varepsilon_1 + \varepsilon_3) + (\varepsilon_1 + \varepsilon_4)$. In the case $i = 3$ these conditions are true for Δ_1 as: $\chi = (\varepsilon_1 + \varepsilon_4) + 10(\varepsilon_2 - \varepsilon_3) + 19(\varepsilon_1 + \varepsilon_3)$, and for the system Δ_2 as: $\chi = 20\varepsilon_1 + 9(\varepsilon_2 + \varepsilon_3) + (\varepsilon_2 + \varepsilon_4)$

Subtract from the first expression the same expression for the root s_{α_i} so we get $(w_0\mu_0 + \mu_1) - (w_0\mu_0 + s_{\alpha_i}\mu_1) = \frac{2(\alpha_i, \mu_1)}{(\alpha_i, \alpha_i)}\alpha_i \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$ for $i = 1, 2, 3, 4$. As $\alpha_i \notin \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$ for $i = 1, 2, 4$ we have $(\alpha_i, \mu_1) = 0$. That implies $\mu_1 = t\pi_3$. The similar expression for the element s_{α_3} doesn't impose any conditions since $\alpha_3 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$. Let us show that $\mu_1 = 0$. Then we would have that all the conditions imposed by the semistable orbits would be the consequences of the conditions imposed by the semistable orbits of the open cell i.e. $w_0\mu_0 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$. Indeed if $\mu_1 = 0$, the conditions for $w \in W_\chi^{st}$ can be rewritten in the form $w_0\mu_0 \in \bigcap \langle \Delta_i \rangle$, where $\chi \in \Delta_i$, but all these conditions are the consequences of the conditions for the open cell, since the intersection is minimal for that. So we have $\text{rk}(\text{Pic}(X_{L_\chi}^{ss} // T)) = 2$.

To prove the equality $\mu_1 = 0$ consider the element $w = s_{\alpha_3}s_{\alpha_4}$. First we prove that ww_0 is χ -semistable, and that it imposes the condition $ww_0\mu_0 + \mu_1 \in \langle w\Delta_1 \rangle \cap \langle w\Delta_2 \rangle$ on μ . It is necessary and sufficient that we have $w\chi \in \sum_{\alpha_j \in (w\Delta_1 \cap w\Delta^+ \cap \Delta^+)} \mathbb{Q}_+\alpha_j$ and the similar condition for the root system Δ_2 . Apply w^{-1} to this expression. Taking into account that w translates the roots $(\varepsilon_3 + \varepsilon_4)$ and ε_4 into negative ones we get $\chi \in \sum_{\alpha_j \in (\Delta_1 \cap \Delta^+ \cap s_{\alpha_4}s_{\alpha_3}\Delta^+)} \mathbb{Q}_+\alpha_j = \sum_{\alpha_j \in (\Delta_1^+ \setminus \{\alpha_4, \varepsilon_3 + \varepsilon_4\})} \mathbb{Q}_+\alpha_j$ and the similar condition for the system Δ_2 . But both conditions are evidently satisfied since ε_4 and $\varepsilon_3 + \varepsilon_4$ do not belong to either of the systems Δ_i . So the conditions imposed by the semistable orbits in the cell Bww_0B/B can be rewritten as $w_0\mu_0 + w^{-1}\mu_1 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$. Taking into account that $\mu_1 = t\pi_3$ we get $w_0\mu_0 + w^{-1}\mu_1 = w_0\mu_0 + ts_{\alpha_4}(\pi_3 - \alpha_3) = w_0\mu_0 + t\pi_3 - t(\varepsilon_3 + \varepsilon_4) \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$. Subtract $w_0\mu_0 + t\pi_3 \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$ from the previous expression. So we get $t(\varepsilon_3 + \varepsilon_4) \in \langle \Delta_1 \rangle \cap \langle \Delta_2 \rangle$, that is possible only when $t = 0$. Consequently $\mu_1 = 0$.

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